A Cartesian Bicategory of Nondeterministic Arrows between Domains

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Abstract

We show how a simple intuitive conception of nondeterministic computation leads to a construction, given a locally ordered bicategory **D** having finite bicategorical products (for example, a bicategory of domains and continuous functions with extensionally ordered homs), of a bicategory $\mathcal{N}\mathbf{D}$ of "nondeterministic arrows" that embeds **D** as a locally full sub-bicategory. The cartesian product on **D** extends to a pseudofunctorial tensor product on $\mathcal{N}\mathbf{D}$. We show that, in case the homs of **D** are bounded-complete directed complete partial orders with composition respecting local directed colimits, then a nondeterministic arrow in $\mathcal{N}\mathbf{D}$ is a left adjoint (*i.e.* a "map") if and only if it is isomorphic to a strict, sup-preserving arrow of **D**. Under the additional assumption that **D** has local terminal objects (*i.e* each hom has a "top"), then $\mathcal{N}\mathbf{D}$ is a cartesian bicategory in the sense of Carboni, *et al.* In addition, we note that the "trace" that exists on **D** via the fixed point theorem extends in a natural way to $\mathcal{N}\mathbf{D}$, thus pointing the way to the use of $\mathcal{N}\mathbf{D}$ for defining the semantics of nondeterministic programming constructs.

Keywords: cartesian bicategory, denotational semantics, domain, nondeterminism, trace

1 Introduction

Traditional denotational semantics takes place in a category **D** of "domains," which typically has as objects some kind of complete partially ordered sets (cpos), with continuous functions as morphisms. The category **D** comes with sufficient structure to permit the construction of various kinds of domains to be used as the targets of semantic maps. Such structure typically includes finite products, various kinds of lifting and sum constructions, construction of function spaces, and the ability to solve recursive domain equations as well as to solve recursive specifications on an individual domain [4]. Such a setting has been very successful as a framework for defining the meaning of deterministic, "function-like" programs, which perform a unique computation on each input. It has not been so successful at defining the meaning of nondeterministic programs, which have the potential to perform many different computations on a single input.

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The traditional approach to defining the meaning of a nondeterministic program P taking inputs in domain A and producing outputs in domain B is to define the meaning of P to be a function $f : A \to \mathcal{P}(B)$, where $\mathcal{P}(\text{ - })$ is a powerdomain construction on $\mathbf{D}$ that has been devised so as to be compatible with the other available constructions [8]. Although this approach is adequate for defining the meanings of first-order nondeterministic programs on "flat" domains, it is not as satisfactory for higher-order programs or programs that operate on structured domains. The requirements imposed by the setting of cpos and continuous functions seem somehow at odds with the desire to describe programs with multiple execution paths. In essence, a shoehorn is used to cast non-functional programs in a functional mold.

An alternative way of thinking about nondeterministic programs is in terms of relations, rather than functions. This approach also works very well for first-order programs on non-structured domains. However, an attempt to extend relational semantics to programs whose inputs and outputs are in non-flat domains immediately reveals serious problems. Dataflow networks (*i.e.* networks of stream-processing programs) [7] make a good test case. In such networks, each individual program, running asynchronously, reads data "tokens" from input streams and emits tokens to output streams. As is well-known, if the individual programs are deterministic in the sense of having functional input/output behavior, then the entire network is as well, and the behavior of the network is determined by the behaviors of the components according to a least fixed point principle (the so-called "Kahn principle"). This is handled quite well in a traditional denotational semantic setting using domains of finite and infinite sequences to model the input and output streams. If one introduces non-functional programs, though, such as programs capable of nondeterministically merging tokens arriving on multiple input streams into a single output stream, then naive attempts to apply denotational semantics fail to assign meanings that agree with the intuitive operational model. Such programs can no longer be modeled in a direct way as continuous functions, attempts to shoehorn them into the functional mold by defining their meanings as continuous functions into a powerdomain produces "unrealistic" infinite results, and attempting to model nondeterministic stream-processing programs as ordinary relations does not even lead to a compositional semantics [1].

If one considers further the failure of powerdomains and ordinary relations to provide a proper semantics for nondeterministic programs, a conclusion one can reach is that it is not enough to simply keep track, as ordinary relations do, of which inputs are related to which outputs; it is necessary to incorporate information about the *ways* in which inputs are related to outputs. This idea suggests attempting to view nondeterministic programs as denoting *generalized* relations, which permit a given input and output to be related in more than one way and to even permit the set of ways of relating an input and output to have algebraic structure, rather than just being a simple set. This kind of approach has been investigated in the context of non-deterministic dataflow [5,9,11,12,13].

In recent years, there has been a great deal of progress in the use of category theory to study theories of generalized relations. One appealing approach looks at a relation r from A to B as an arrow (1-cell) in a *bicategory*, whose 2-cells generalize the inclusion relations that hold between ordinary set-theoretic relations.

The theory of "cartesian bicategories" developed by Carboni, Walters, and their colleagues [2,3] is an elegant example of the study of relations from this point of view. It is significant that traditional categories of domains, equipped as they are with an ordering on their homs, are *already* bicategories.

If one accepts the ideas: (1) that nondeterministic programs should be treated as generalized relations; and (2) that generalized relations occur as the 1-cells in some kind of a bicategory, then it becomes natural to look for a bicategorical setting that generalizes the classical approach based on cpos, continuous functions, and least fixed points. In particular, one would like to have a bicategory that embeds as a sub-bicategory a category of cpos and continuous functions and in addition has arrows representing generalized relations that can be used to define the semantics of nondeterministic programs.

To explore a bit further, suppose that we are working with a particular category $\mathbf{D}$ of "domains." We don't need to know exactly what "domains" or their morphisms are, but we do assume that $\mathbf{D}$ has partially ordered homs and that composition respects the ordering. That is, we assume that $\mathbf{D}$ is a locally ordered bicategory. In addition, we assume that $\mathbf{D}$ has a terminal object I and finite products (denoted by $\times$), both in the bicategorical sense, which implies that each hom-category $\mathbf{D}(X, I)$ is equivalent to the one-object, one-arrow category, and each hom-category $\mathbf{D}(X, A \times B)$ is equivalent to the product category $\mathbf{D}(X, A) \times \mathbf{D}(X, B)$. As a motivating example, $\mathbf{D}$ could be the category of bounded-complete directed-complete partial orders (dcpos) with continuous functions as arrows, though we don't need to make use of bounded-completeness or continuity just yet.

Now, suppose we want to extend $\mathbf{D}$ from a bicategory of "function-like" arrows to a bicategory $\mathcal{N}\mathbf{D}$ of "relation-like" arrows that can serve as denotations for nondeterministic programs. Our intention is to keep the same objects but expand the collection of arrows (as well as the 2-cells). The question is, where are going to use for these new, "nondeterministic arrows"? One idea is to think of nondeterministic arrows as functions that produce sets of results, but that leads to the powerdomain approach with its attendant difficulties. Another idea is to think of a nondeterministic arrow from A to B as a *span* $A \leftarrow U \rightarrow B$ in $\mathbf{D}$, as spans nicely capture the idea of generalized relations that permit inputs to be related to outputs in multiple ways, possibly with structure. However, we are then faced with the problems of trying to determine precisely *which* spans we should use (*i.e* how generalized should our generalized relations be?) and how they should compose (the composition of general spans by pullback does not immediately lead to a way of nicely embedding $\mathbf{D}$ into our new bicategory). Instead of doing this, we are going to describe a different approach, which leads in the end to a result that could also have been described in terms of spans, but which provides solutions to the above problems more automatically.

Our approach is based on a key revision to the way we think about how nondeterminism is related to computation: instead of thinking of nondeterminism as presenting choices that unfold from time to time as computation progresses, we think of a complete possible resolution of the nondeterministic choices as being supplied in advance, essentially as an additional input. This is much like the idea of an "oracle," except that, unlike the usual input-only conception of an oracle, our nondeterministic arrows will also *output* this kind of information as well as input it. In more detail, we think of a nondeterministic program P taking inputs in A and producing outputs in B as a function $p : A \times U \to U \times B$, where U is an "object of computations". Such a function takes an input $a \in A$ and a *target computation* $u \in U$ and produces, in addition to a result $b \in B$, a computation $u' \in U$ that constitutes an *accessible approximation* of the given computation u. The intuition is that u' represents that portion of the given target computation that it is possible for P to perform, given that the input available is a. By assuming that a target computation u is given in advance, and that computation proceeds toward this target computation, we avoid failures of continuity that arise when we try to think of the nondeterministic choices as being resolved incrementally as the computation unfolds.

Some conditions on p are necessary in order to accurately capture the above intuition. In particular, if p takes $\langle a, u\rangle$ to $\langle u', b\rangle$, then if u' is to be regarded as an accessible approximation to target computation u it should be the case that $u' \sqsubseteq u$. In addition, the notion of accessible approximation ought in some sense to be "stable": if u' is the approximation that can be reached, given input a, to target computation u, then u' ought also to be the approximation that can be reached, given input a, if the target computation is u' itself. We are therefore led to make the following definition.

Definition 1.1 [Nondeterministic Arrow] Let A and B be objects of $\mathbf{D}$. A *nondeterministic arrow* from A to B consists of an ordinary arrow $p : A \times U \to U \times B$ such that the following conditions hold:

(i) $\pi^U_{U,B} \cdot p \sqsubseteq \pi^U_{A,U}$.

(ii) $p \cdot \langle \pi^A_{A,U}, \pi^U_{U,B} \cdot p\rangle = p$.

Here $\pi^A_{A,U} : A \times U \to A$ and $\pi^U_{U,B} : U \times B \to B$ denote projections, and $\cdot$ denotes composition. We refer to U as the *object of computations* of p, and we write $p : A \xrightarrow[U]{} B$ to assert that p is a nondeterministic arrow from A to B, with object of computations U.

At this point and in the sequel it is extremely useful to visualize expressions denoting arrows of $\mathbf{D}$ as "schematic diagrams." For example, condition (ii) of Definition 1.1 can be visualized as shown in Fig. 1. The properties of $\mathbf{D}$ (in particular the fact that it is a cartesian category) enable us to perform sound reasoning using these schematic diagrams rather than having to work with complex terms or even traditional commutative diagrams.

Note that the concept of nondeterministic arrow generalizes that of ordinary arrows, in the sense that an ordinary arrow $f : A \to B$ can be identified with the nondeterministic arrow $f : A \times I \to I \times B$, where I is the terminal object in $\mathbf{D}$ (*i.e.* the one-point domain). Aside from placing a subscript under the arrow, we won't bother to make any notational distinction between an ordinary arrow $f : A \to B$ and the corresponding nondeterministic arrow $f : A \xrightarrow[I]{} B$.

Nondeterministic arrows $p : A \xrightarrow[U]{} B$ and $q : B \xrightarrow[V]{} C$ can be composed in a way that generalizes composition of ordinary arrows. The intuition is to think of p and q as each working separately toward their respective target computations, except

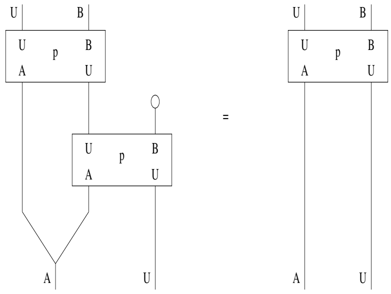

Fig. 1. Condition (ii) of Definition 1.1

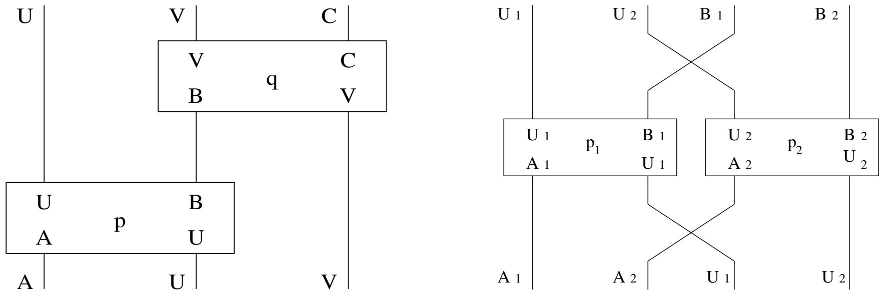

Fig. 2. Composite and Tensor Product of Nondeterministic Arrows

that the output produced by p is consumed by q.

Definition 1.2 [Composition] Suppose $p : A \underset{U}{\longrightarrow} B$ and $q : B \underset{V}{\longrightarrow} C$ are nondeterministic arrows. The *composite* of p and q is the nondeterministic arrow $q \circ p : A \underset{U \times V}{\longrightarrow} C$ defined by:

$$q \circ p \quad = \quad (1_U \times q) \cdot (p \times 1_V)$$

(see Fig. 2).

Note that ordinary identity arrows $1_A : A \to A$ determine nondeterministic arrows $1_A : A \underset{I}{\longrightarrow} A$ that are "almost" left and right units for the generalized notion of composition, except that if $p : A \underset{U}{\longrightarrow} V$ then $1_B \circ p : A \underset{U \times I}{\longrightarrow} B$ and $p \circ 1_A : A \underset{I \times U}{\longrightarrow} B$. But $I \times U \simeq U \simeq U \times I$, so in some sense p, $1_B \circ p$, and $p \circ 1_A$ are "isomorphic," but this has yet to be made precise. Similarly, composition of nondeterministic arrows is "almost" associative: if $p : A \underset{U}{\longrightarrow} B$, $q : B \underset{V}{\longrightarrow} C$, and $r : C \underset{W}{\longrightarrow} D$, then $r \circ (q \circ p) : A \underset{(U \times V) \times W}{\longrightarrow} D$, whereas $(r \circ q) \circ p : A \underset{U \times (V \times W)}{\longrightarrow} D$. In any case, the situation in which unit and associativity laws hold only up to isomorphism is characteristic of a bicategory.

The category **D** has a cartesian structure derived from the terminal object I and products $\times$. As is well-known, this structure can be axiomatized in terms of equations between terms constructed using identities ($1_A : A \to A$), diagonals ($\delta_A : A \to A \times A$), terminals ($\tau_A : A \to I$), symmetries ($\sigma_{A,B} : A \times B \to B \times A$), left and right unit isomorphisms ($\lambda_A : I \times A \simeq A$ and $\rho_A : A \times I \simeq A$), and associativities ($\alpha_{A,B,C} : (A \times B) \times C \simeq A \times (B \times C)$ and their inverses $\alpha^{-1}_{A,B,C}$), using $\cdot$ and $\times$.

Pairing and projections can then be recovered from this structure via the definitions

$$\langle f, g\rangle = (f \times g) \cdot \delta_X \qquad \pi^A_{A,B} = \rho_A \cdot (1_A \times \tau_B) \qquad \pi^B_{A,B} = \lambda_A \cdot (\tau_A \times 1_B)$$

Since it is very tedious to keep track of associativity and left and right unit isomorphisms, it will be convenient for us to assume that these are in fact identities (*i.e.* that the monoidal structure determined by the product is *strict*). There is no real harm in this assumption, since it can always be arranged by a suitable choice of products. So we will generally not bother about differences in bracketing of products, and when there is no loss in clarity we will also drop factors of I in products.

Since all of the above arrows are also nondeterministic arrows, and equational laws that hold between ordinary arrows clearly also hold up to isomorphism between nondeterministic arrows, the cartesian structure on $\mathbf{D}$ will induce (assuming that the necessary coherence conditions hold) a symmetric monoidal structure (and more) on $\mathcal{N}\mathbf{D}$, if we can define a suitable generalization $\otimes$ of the product $\times$ on arrows of $\mathbf{D}$. This we do as follows:

Definition 1.3 [Tensor Product] Suppose $p : A_1 \underset{U_1}{\longrightarrow} A_2$ and $q : A_2 \underset{U_2}{\longrightarrow} B_2$ are nondeterministic arrows. The *tensor product* of p and q is the nondeterministic arrow $p \otimes q : A_1 \times A_2 \underset{U_1 \times U_2}{\longrightarrow} B_1 \times B_2$ defined by:

$$p \otimes q \;\; = \;\; (1_{U_1} \times \sigma_{B_1,U_2} \times 1_{B_2}) \cdot (p \times q) \cdot (1_{A_1} \times \sigma_{A_2,U_1} \times 1_{U_2})$$

(see Fig. 2).

We now wish to show that nondeterministic arrows are the 1-cells of a bicategory $\mathcal{N}\mathbf{D}$ that embeds $\mathbf{D}$, but in order to do so we have to find the right notion of 2-cell; *i.e.* of *morphism of nondeterministic arrows*. To motivate the definition, we need to explore in a bit more detail the intuition behind our definition of nondeterministic arrow.

2 Morphisms

Suppose $p : A \underset{U}{\longrightarrow} B$ is a nondeterministic arrow. As an ordinary arrow, $p : A \times U \to U \times B$, hence p can be written as $\langle p^U, p^B\rangle$ where $p^U : A \times U \to U$ and $p^B : A \times U \to B$. In addition, p determines, for each $a \in A$, an arrow $p^U_a : U \to U$, defined by $p^U_a(u) = p^U(\langle a, u\rangle)$. From Definition 1.1 it follows that p^U_a is *decreasing* ($p^U_a \sqsubseteq 1_U$) and *idempotent* ($p^U_a \cdot p^U_a = p^U_a$). It is therefore a *coreflexive* (or "coclosure").

Let us temporarily assume that the $\mathbf{D}$ is in fact the category of bounded complete dcpos and continuous functions. Then the image $\mathrm{Im}(p^U_a)$ of p^U_a (which also happens to be the set $\{u : p^U_a(u) = u\}$ of fixed points of p^U_a) is a coreflective subobject of U (in domain-theoretic terminology this would be called a "normal subdomain" or "retract" [4]). We will refer to $\mathrm{Im}(p^U_a)$ as the *fiber of p over a*, suggesting a connection with fibrations which is in fact the motivation for our definition of morphism. If u is an element of the fiber of p over a, then u is also an element of the fiber of p over a' for any a' with $a \sqsubseteq a'$. This is because $p_a^U(u) = u$, hence $u \sqsubseteq p_{a'}^U(u)$ by monotonicity, but also $p_{a'}^U(u) \sqsubseteq u$ because $p_{a'}^U$ is decreasing. Conversely, if u' is the fiber of p over a', then for each $a \in A$ such that $a \sqsubseteq a'$ the fiber of p over a' contains a greatest $u \sqsubseteq u'$ such that u is in the fiber of p over a.

For our specific choice of $\mathbf{D}$, then, a nondeterministic arrow $p : A \underset{U}{\longrightarrow} B$ therefore determines an A-indexed collection $\{U_a : a \in A\}$ of coreflective subdomains of U, such that whenever $a \sqsubseteq a'$ then the domain U_a is a coreflective subdomain of $U_{a'}$. Each subdomain U_a consists of those computations $u \in U$ that are accessible given input $a \in A$. In addition, p determines, for each $a \in A$, an *output map* $p_a^B : U_a \to B$. By the monotonicity of p we have the following relations:

$$a \sqsubseteq a' \qquad \text{implies} \qquad p_a^B(u) \sqsubseteq p_{a'}^B(u), \text{ for all } u \in U_a.$$

$$a \sqsubseteq a' \qquad \text{implies} \qquad p_a^B(p_a^U(u)) \sqsubseteq p_{a'}^B(u), \text{ for all } u \in U_{a'}.$$

That is, the embeddings and reflectors between the fibers preserve output in a *lax* sense. To summarize the above discussion, a nondeterministic arrow $p : A \underset{U}{\longrightarrow} B$ determines a *functor* $\hat{p}$ from A to the category of B-labeled coreflective subdomains of U, where the latter is equipped with morphisms that are lax output-preserving embeddings.

The above structure is in accordance with our intuition about nondeterministic arrows. When presented with input $a \in A$, a nondeterministic program can proceed toward any target computation u in the fiber U_a. Associated with each such computation is a corresponding output. If the input should increase from a to a', then additional computations in $U_{a'}$ become available, along with additional output. The *maximal* elements of U_a represent possible computations on input a that are fully developed, or "completed."

We now present our definition of "morphism of nondeterministic arrow," which is intended to capture that the above structure of inputs, fibers, and output maps is preserved in a suitable way. At this point we drop our temporary assumption about the concrete structure of $\mathcal{D}$, as it is not required in order to state the definition.

Definition 2.1 Suppose $p : A \underset{U}{\longrightarrow} B$ and $q : A \underset{V}{\longrightarrow} B$ are nondeterministic arrows. A *morphism* from p to q is an arrow $\mu : A \times U \to V$ of $\mathbf{D}$ that satisfies the following conditions (see Fig. 3):

(i) $q^U \cdot \langle \pi_{A,U}^A, \mu \rangle = \mu$.

(ii) $\mu \cdot \langle \pi_{A,U}^A, p^U \rangle = \mu$.

(iii) $p^B \sqsubseteq q^B \cdot \langle \pi_{A,U}^A, \mu \rangle$.

We write $\mu : p \Rightarrow q$ to assert that μ is a morphism from p to q.

It is convenient to introduce one further bit of notation: if $f : A \times U \to V$ is any arrow, then $f_{\#_A}$ denotes the arrow $\langle \pi_{A,U}^A, f \rangle : A \times U \to A \times V$. Using this notation, together with the abbreviations $p^U = \pi_{U,B}^U \cdot p$ and $p^B = \pi_{U,B}^B \cdot p$, we can write the above conditions in the less-cluttered form: (i) $q^U \cdot \mu_{\#_A} = \mu$; (ii) $\mu \cdot {p^U}_{\#_A} = \mu$; and (iii) $p^B \sqsubseteq q^B \cdot \mu_{\#_A}$.

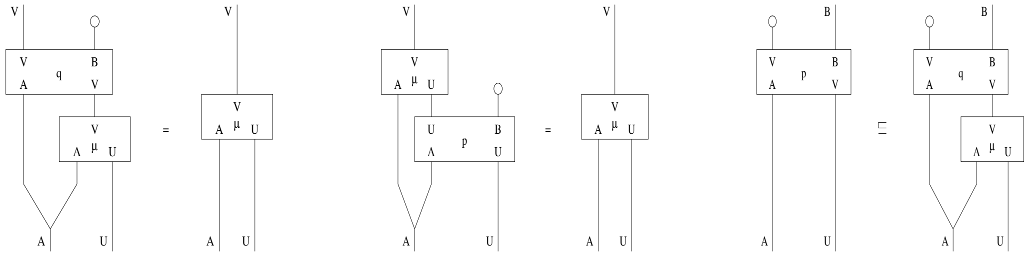

Fig. 3. Morphism Conditions

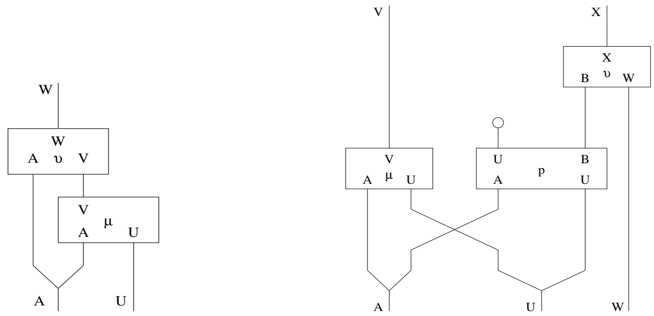

Fig. 4. Vertical and Horizontal Composite of Morphisms

In the context of our intuitive motivation, conditions (i) and (ii) express the idea that μ essentially "acts fiberwise," so that for each $a \in A$, the mapping $\mu\langle a, \text{-}\rangle$ is the least extension to all of U of a mapping that takes the fiber of p over a to the fiber of q over a. Condition (iii) expresses the idea that a morphism preserves output in a lax fashion.

Proposition 2.2 *Suppose $f, g : A \xrightarrow[I]{} B$ are nondeterministic arrows corresponding to the ordinary arrows $f, g : A \to B$. Then $f \sqsubseteq g$ in $\mathbf{D}$ if and only if there exists a morphism $\mu : f \Rightarrow g$. Moreover, if such a morphism μ exists, then it is unique.*

Note that the only possible choice for an arrow $\mu : A \times I \to I$ is the terminal arrow $\tau_{A\times I}$. For such an arrow, conditions (i) and (ii) in Definition 2.1 are trivial. Condition (iii) is satisfied if and only if $f \sqsubseteq g$.

Definition 2.3 [Composition, Identities] Suppose $p : A \xrightarrow[U]{} B$, $q : A \xrightarrow[V]{} B$, and $r : A \xrightarrow[W]{} B$ are nondeterministic arrows. If $\mu : p \Rightarrow q$ and $\nu : q \Rightarrow r$, then the (vertical) *composite* of μ and ν is the morphism $\nu * \mu : p \Rightarrow r$ defined by $\nu * \mu = \nu \cdot \mu_{\#_A}$ (see Fig. 4). The *identity morphism* on p is the morphism $1_p : p \Rightarrow p$ corresponding to the arrow $p^U : A \times U \to U$.

Proposition 2.4 *Composition of morphisms is associative, with identity morphisms as units. Thus for each pair of objects (A, B) the set $\mathcal{N}\mathbf{D}(A, B)$ of nondeterministic arrows from A to B, equipped with the morphisms between them and the specified identities, is a category.*

As usual, an *isomorphism* from p to q is a morphism $\mu : p \Rightarrow q$ that is invertible. We say that p and q are *isomorphic* if there exists an isomorphism $\mu : p \Rightarrow q$. Note that, because of the choice we have made for identity morphisms, nondeterministic arrows $p : A \xrightarrow[U]{} B$ and $q : A \xrightarrow[V]{} B$ can be isomorphic in $\mathcal{N}\mathbf{D}$ without necessarily having U and V be isomorphic objects of $\mathbf{D}$.

Proposition 2.5 *Suppose $p : A \xrightarrow[U]{} B$ and $q : B \xrightarrow[V]{} U$ are nondeterministic arrows. Then arrow $\mu : A \times U \to V$ of $\mathbf{D}$ determines an isomorphism $\mu : p \simeq q$ if and only if there exists an arrow $\nu : A \times V \to U$ of $\mathbf{D}$ such that $\nu \cdot \mu_{\#_A} = p^U$ and $\mu \cdot \nu_{\#_A} = q^V$ hold in $\mathbf{D}$.*

In order for nondeterministic arrows to be the 1-cells of a bicategory, with morphisms as the 2-cells, we need composition of nondeterministic arrows to be *functorial*; *i.e.* for each triple of objects (A, B, C) sequential composition should determine a functor

$$\circ : \mathcal{N}\mathbf{D}(B, C) \times \mathcal{N}\mathbf{D}(A, B) \to \mathcal{N}\mathbf{D}(A, C).$$

Such a functor will need to produce, given morphisms $\mu : p \Rightarrow q$ and $\nu : r \Rightarrow s$ of nondeterministic arrows $p, q \in \mathcal{N}\mathbf{D}(A, B)$ and $r, s \in \mathcal{N}\mathbf{D}(B, C)$ a *horizontal composite*

$$\nu \circ \mu : r \circ p \Rightarrow s \circ q.$$

Definition 2.6 Suppose $p : A \xrightarrow[U]{} B$, $q : A \xrightarrow[V]{} B$, $r : B \xrightarrow[W]{} C$, and $s : B \xrightarrow[X]{} C$ are nondeterministic arrows, and $\mu : p \Rightarrow q$ and $\nu : r \Rightarrow s$ are morphisms. The *horizontal composite* of μ and ν is the morphism $\nu \circ \mu : r \circ p \Rightarrow s \circ q$ defined as follows (see Fig. 4):

$$\nu \circ \mu \quad = \quad (1_V \times \nu) \cdot (\mu \times p^B \times 1_W) \cdot (\delta_{A \times U} \times 1_W)$$

It is not immediately obvious that this definition of horizontal composite works properly. In order to verify that the horizontal composite of morphisms is again a morphism requires some key observations.

Lemma 2.7 *Suppose $p : A \xrightarrow[U]{} B$ is a nondeterministic arrow. Then $u : X \to U$ and $a \sqsubseteq a' : X \to A$ implies $p^U \cdot \langle a', p^U \cdot \langle a, u \rangle\rangle \quad = \quad p^U \cdot \langle a, u \rangle$.*

Corollary 2.8 *Suppose $p : A \xrightarrow[U]{} B$ and $q : A \xrightarrow[V]{} B$ are nondeterministic arrows, and $\mu : p \Rightarrow q$ is a morphism. Then $u : X \to U$ and $a \sqsubseteq a' : X \to A$ implies*

$$q^V \cdot \langle a', \mu \cdot \langle a, u \rangle\rangle \quad = \quad q^V \cdot \langle a, \mu \cdot \langle a, u \rangle\rangle$$

Corollary 2.9 *$\nu \circ \mu$ satisfies condition (i) of Definition 2.1.*

Proposition 2.10 *The horizontal composite of morphisms is again a morphism, the horizontal composite of identity morphisms is again an identity morphism, and horizontal composite satisfies the interchange law: $(\rho * \pi) \circ (\nu * \mu) = (\rho \circ \nu) * (\pi \circ \mu)$. Composition of nondeterministic arrows therefore determines a family of functors:*

$$\circ : \mathcal{N}\mathbf{D}(B, C) \times \mathcal{N}\mathbf{D}(A, B) \to \mathcal{N}\mathbf{D}(A, C).$$

The interchange law can be verified by comparing the schematic diagrams for $(\rho \circ \nu) * (\pi \circ \mu)$, and $(\rho * \pi) \circ (\nu * \mu)$, using Corollary 2.9 and making an additional application of Corollary 2.8.

We can now establish:

Theorem 2.11 *There is a bicategory $\mathcal{N}\mathbf{D}$ having the objects of $\mathbf{D}$ as 0-cells, the nondeterministic arrows as 1-cells, and the morphisms of nondeterministic arrows as 2-cells, with the indicated identities and composition, and evident associativity and unit isomorphisms. The correspondence between ordinary arrows $f : A \to B$ and nondeterministic arrows $f : A \underset{I}{\longrightarrow} B$ extends to an equivalence of $\mathbf{D}$ with a locally full sub-bicategory of $\mathcal{N}\mathbf{D}$.*

3 Maps

Recall that an *adjunction* in a bicategory consists of a pair of 1-cells $g : A \to B$ (the left adjoint) and $f : B \to A$ (the right adjoint), together with 2-cells $\eta : 1_A \Rightarrow f \cdot g$ (the *unit*) and $\epsilon : g \cdot f \Rightarrow 1_B$ such that the so-called "triangle identities" are satisfied:

$$(\epsilon g) \cdot (g\eta) = 1_g \qquad\qquad (f\epsilon) \cdot (\eta f) = 1_f.$$

Work of Carboni, Walters *et al* on cartesian bicategories [2,3] indicates the important role played by the *maps*, which are arrows that are left adjoints. In particular, in a bicategory whose 1-cells are some kind of generalized relation one expects the maps to be the relations that are in some sense "functional."

Proposition 3.1 *Every map $g : A \underset{V}{\longrightarrow} B$ in $\mathcal{N}\mathbf{D}$ is isomorphic to an arrow of the form $\hat{g} : A \underset{I}{\longrightarrow} B$.*

Proposition 3.2 *Suppose $g : A \to B$ is an arrow of $\mathbf{D}$ and $g : A \underset{I}{\longrightarrow} B$ is the corresponding nondeterministic arrow in $\mathcal{N}\mathbf{D}$. If g has a right adjoint $f : B \to A$ in $\mathbf{D}$, then $g : A \underset{I}{\longrightarrow} B$ has $f : B \underset{I}{\longrightarrow} A$ as a right adjoint in $\mathcal{N}\mathbf{D}$.*

Proposition 3.3 *Suppose $g : A \to B$ is an arrow of $\mathbf{D}$. If $\langle g, 1_A\rangle : A \to B \times A$ has a right adjoint left inverse $f : B \times A \to A$ in $\mathbf{D}$, then $g : A \underset{I}{\longrightarrow} B$ is a map in $\mathcal{N}\mathbf{D}$, with right adjoint $g^* : B \underset{A}{\longrightarrow} A$ given by the arrow $\delta_A \cdot f : B \times A \to A \times A$ of $\mathbf{D}$.*

Corollary 3.4 *Each nondeterministic arrow $\tau_A : A \underset{I}{\longrightarrow} I$ is a map in $\mathcal{N}\mathbf{D}$, with right adjoint $\tau_A^* : I \underset{A}{\longrightarrow} A$ given by the arrow $\delta_A \cdot \pi^A_{I,A} : I \times A \to A \times A$ of $\mathbf{D}$.*

We can further characterize the maps in $\mathcal{N}\mathbf{D}$ if we impose additional conditions on $\mathbf{D}$.

Assumption 3.5 *Suppose $\mathbf{D}$ has homs that are bounded complete dcpos with least element, such that composition preserves directed suprema.*

Proposition 3.6 *Under Assumption 3.5, a nondeterministic arrow $g : A \underset{V}{\longrightarrow} B$ in $\mathcal{N}\mathbf{D}$ is a map if and only if as an arrow $g : A \times V \to V \times B$ of $\mathbf{D}$ it is strict ($g \cdot \bot_A = \bot_B$) and sup-preserving $g \cdot (f_1 \sqcup f_2) = (g \cdot f_1) \sqcup (g \cdot f_2)$ whenever $f_1 \sqcup f_2$ exists).*

Corollary 3.7 *Under Assumption 3.5, the following are maps in $\mathcal{N}\mathbf{D}$:*

(i) *All nondeterministic arrows $\delta_A : A \underset{I}{\longrightarrow} A \times A$.*

(ii) *All nondeterministic arrows* $\pi^A_{A,B} : A \times B \underset{I}{\longrightarrow} A$ *and* $\pi^B_{A,B} : A \times B \underset{I}{\longrightarrow} B$ *determined by projections in* $\mathbf{D}$.

The right adjoint to $\delta_A : A \underset{I}{\longrightarrow} A \times A$ is the arrow $\delta^*_A : A \times A \underset{I}{\longrightarrow} A$ corresponding to the arrow $\delta^*_A : A \times A \to A$ which is right adjoint to δ_A in $\mathbf{D}$. If $f_1, f_2 : X \to A$, then $\delta^*_A \cdot \langle f_1, f_2 \rangle$ is the meet $f_1 \sqcap f_2$ of f_1 and f_2 in $\mathbf{D}(X, A)$. A right adjoint to $\pi^A_{A,B} : A \times B \underset{I}{\longrightarrow} A$ is the nondeterministic arrow $A \underset{B}{\longrightarrow} A \times B$ corresponding to the arrow

$$(\sigma_{B,A} \times 1_B) \cdot (1_A \times \delta_B) : A \times B \to B \times A \times B$$

of $\mathbf{D}$. Intuitively, this nondeterministic arrow takes its argument a and pairs it with a nondeterministically chosen element of B to produce its output.

4 Cartesian Structure

In Definition 1.3 we defined a binary operation $\otimes$ that takes nondeterministic arrows $p_1 : A_1 \underset{U_1}{\longrightarrow} B_1$ and $p_2 : A_2 \underset{U_2}{\longrightarrow} B_2$ to a nondeterministic arrow $p_1 \otimes p_2 : A_1 \times A_2 \underset{U_1 \times U_2}{\longrightarrow} B_1 \times B_2$. This operation can be extended to morphisms. Suppose $p_1 : A_1 \underset{U_1}{\longrightarrow} B_1$, $q_1 : A_1 \underset{V_1}{\longrightarrow} B_1$, $p_2 : A_2 \underset{U_2}{\longrightarrow} B_2$, and $q_2 : A_2 \underset{V_2}{\longrightarrow} B_2$ and suppose $\mu_1 : p_1 \Rightarrow q_1$ and $\mu_2 : p_2 \Rightarrow q_2$. Let $\mu_1 \otimes \mu_2$ be the arrow

$$\mu_1 \otimes \mu_2 : A_1 \times A_2 \times U_1 \times U_2 \to V_1 \times V_2$$

defined by

$$\mu_1 \otimes \mu_2 \quad = \quad (\mu_1 \times \mu_2) \cdot (1_{A_1} \times \sigma_{A_2,U_1} \times 1_{U_2}).$$

It is then straightforward to verify that $\mu_1 \otimes \mu_2$ satisfies the conditions for a morphism $\mu_1 \otimes \mu_2 : p_1 \otimes p_2 \Rightarrow q_1 \otimes q_2$, that $1_{p_1} \otimes 1_{p_2} = 1_{p_1 \otimes p_2}$, and that $(\nu_1 * \mu_1) \otimes (\nu_2 * \mu_2) = (\nu_1 \otimes \nu_2) * (\mu_1 \otimes \mu_2)$, so that $\otimes$ determines a family of functors

$$\otimes : \mathcal{N}\mathbf{D}(A_1, B_1) \times \mathcal{N}\mathbf{D}(A_2, B_2) \to \mathcal{N}\mathbf{D}(A_1 \times A_2, B_1 \times B_2).$$

For each pair of objects (A_1, A_2) there is an isomorphism $\phi : 1_{A_1 \times A_2} \simeq 1_{A_1} \otimes 1_{A_2}$. In addition, given pairs of objects (A_1, A_2), (B_1, B_2), and (C_1, C_2), and pairs of nondeterministic arrows $(p_1, p_2) : (A_1, A_2) \to (B_1, B_2)$, and $(q_1, q_2) : (B_1, B_2) \to (C_1, C_2)$, there is an evident isomorphism $\psi : (q_1 \otimes q_2) \circ (p_1 \otimes p_2) \simeq (q_1 \circ p_1) \otimes (q_2 \circ p_2)$, and the isomorphisms ψ can be shown to be the components of natural transformations that behave sensibly with respect to associativity and unit isomorphisms of $\mathcal{N}\mathbf{D}$. This is somewhat messy, because the isomorphisms involved are not simply isomorphisms in $\mathbf{D}$, but rather are constructed from identities between nondeterministic arrows. However, if $p \to q$ is any of these isomorphisms, then it can be expressed in a normal form $q^V \cdot (1_A \times \gamma)$, where $\gamma : U \to V$ is a canonical isomorphism in $\mathbf{D}$. Such normal forms are unique due to coherence in $\mathbf{D}$. Moreover, any isomorphism constructed by pasting together isomorphisms in normal form can again be reduced to normal form by permuting the p^U and q^V terms with the isomorphisms in $\mathbf{D}$ and using Definition 1.1 (ii) to cancel out excess p^U and q^V terms that accumulate. We therefore have:

Proposition 4.1 $\otimes$ *extends to a pseudofunctor* $\otimes : \mathcal{N}\mathbf{D} \times \mathcal{N}\mathbf{D} \to \mathcal{N}\mathbf{D}$.

Proposition 4.2 *Under Assumption 3.5, $\mathcal{N}\mathbf{D}$ has local binary products, given by the formula*

$$p \sqcap q \quad \simeq \quad \delta_B^* \circ (p \otimes q) \circ \delta_A$$

which defines, for each pair of objects (A, B) a functor

$$\sqcap : \mathcal{N}\mathbf{D}(A, B) \times \mathcal{N}\mathbf{D}(A, B) \to \mathcal{N}\mathbf{D}(A, B).$$

Tensor product $\otimes$ can be recovered from local product $\sqcap$ in the sense that the formula

$$P(p_1, p_2) \quad = \quad ((\pi^{B_1}_{B_1,B_2})^* \circ p_1 \circ \pi^{A_1}_{A_1,A_2}) \sqcap ((\pi^{B_2}_{B_1,B_2})^* \circ p_2 \circ \pi^{A_2}_{A_1,A_2}) \qquad (1)$$

defines a functor $P : \mathcal{N}\mathbf{D}(A_1, B_1) \times \mathcal{N}\mathbf{D}(A_2, B_2) \to \mathcal{N}\mathbf{D}(A_1 \times A_2, B_1 \times B_2)$ that is naturally isomorphic to $\otimes$.

In the terminology of [2], a bicategory $\mathcal{B}$ is called *precartesian* if the locally full subbicategory of maps has bicategorical products, denoted by $\times$, and biterminal, denoted by I, and in addition each hom-category $\mathcal{B}(A, B)$ has local products, denoted by $\sqcap$, and terminal, denoted by $\top$. In that case, the bicategory $\mathcal{B}$ admits certain lax functors

$$\mathcal{B} \times \mathcal{B} \xrightarrow{\otimes} \mathcal{B} \xleftarrow{I} I$$

derived from the product structure on the subbicategory of maps. The functor $\otimes$ is defined via formula (1) of Proposition 4.2. The functor I selects a monad in $\mathcal{B}$, consisting of the object I (the terminal object in the subbicatgory of maps), the endo-1-cell $\top : I \to I$, equipped with the terminal 2-cells $1_I \Rightarrow \top$ and $\top\top \Rightarrow \top$ as unit and multiplication, respectively. A precartesian bicategory is called *cartesian* if $\otimes$ and I are in fact pseudofunctors [2,14].

Note that in view of Proposition 4.2, the tensor product in $\mathcal{N}\mathbf{D}$ derived via the formula (1) is naturally isomorphic to the tensor product we have defined explicitly. We have already observed (Proposition 4.1) that this tensor product is pseudofunctorial. In addition, the monad in $\mathcal{N}\mathbf{D}$ selected by the functor I defined above is precisely that consisting of the object I, the endo-1-cell $1_I : I \xrightarrow[I]{} I$ and the identity 2-cell on this 1-cell as both unit and multiplication, because in $\mathcal{N}\mathbf{D}$ the identity $1_I : I \to I$ is also the local terminal. We therefore obtain:

Theorem 4.3 *Suppose Assumption 3.5, and in addition suppose that $\mathbf{D}$ has local terminals. Then $\mathcal{N}\mathbf{D}$ is a cartesian bicategory whose locally full subbicategory of maps is equivalent to the locally full subbicategory of $\mathbf{D}$ determined by the strict, sup-preserving arrows.*

It is shown in [2] that every cartesian bicategory is a symmetric monoidal bicategory. Thus we also have:

Corollary 4.4 *Under the conditions of Theorem 4.3, $\mathcal{N}\mathbf{D}$ is a symmetric monoidal bicategory.*

The assumption that $\mathbf{D}$ has local terminals can always be arranged by artificially adding a top element to each of the homs, and suitably extending composition to include it. However, requiring the existence of top elements tends to be somewhat

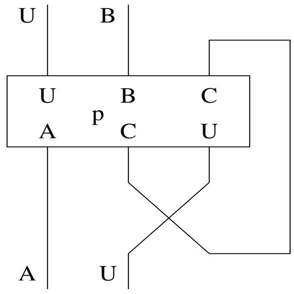

Fig. 5. Trace of a Nondeterministic Arrow

awkward for use in denotational semantics. Since we are able to define tensor product $\otimes$ (and its unit, I) on $\mathcal{N}\mathbf{D}$ without needing to refer to local terminals, it seems clear that $\mathcal{N}\mathbf{D}$ "ought to be" a symmetric monoidal bicategory (though not quite a cartesian bicategory in the sense of [2]) even without the assumption that $\mathbf{D}$ has local terminals.

5 Trace

The category $\mathbf{D}$ is *traced* (in the sense of [6]), which means that it is equipped with a family of functions $\mathrm{T}^C_{A,B} : \mathbf{D}(A \times C, B \times C) \to \mathbf{D}(A, B)$ subject to conditions that express naturality and the idea that $\mathrm{T}^C_{A,B}(f)$ is a fixed point of the "feedback loop" obtained by connecting the C-output of f back to its C-input. We can use the cpo structure on the homs to explicitly define the trace on $\mathbf{D}$. In particular, suppose $f : A \times C \to B \times C$ is an arrow. Define the functor $\Phi : \mathbf{D}(A, B \times C) \to \mathbf{D}(A, B \times C)$ by

$$\Phi(g) \quad = \quad f \cdot (1_A \times (\pi^C_{B,C} \cdot g)) \cdot \delta_A.$$

Let $\mu\Phi$ be the least fixed point of Φ. Then $\mathrm{T}^C_{A,B}(f)$ is defined to be the arrow $\pi^B_{B,C} \cdot (\mu\Phi)$.

The trace $\mathrm{T}^C_{A,B}$ on $\mathbf{D}$ can be extended to a similar operation $\mathcal{T}^C_{A,B}$ on nondeterministic arrows.

Definition 5.1 [Trace] Suppose $p : A \times C \underset{U}{\longrightarrow} B \times C$ is a nondeterministic arrow. The *trace* of p by C is the nondeterministic arrow $\mathcal{T}^C_{A,B}(p) : A \underset{U}{\longrightarrow} B$ defined by:

$$\mathcal{T}^C_{A,B}(p) \quad = \quad \mathrm{T}^C_{A \times U, U \times B}(p \cdot (1_A \times \sigma_{U,C}))$$

(see Fig. 5).

To see whether the above definition is sensible, we need to invoke our original intuition for the definition of nondeterministic arrows. Given a nondeterministic arrow $p : A \times C \underset{U}{\longrightarrow} B \times C$, the nondeterministic arrow $\mathcal{T}^C_{A,B}(p)$ will have the same object of computations as p; *i.e.* $\mathcal{T}^C_{A,B}(p) : A \underset{U}{\longrightarrow} B$. Note that if $U = I$; that is, p is the nondeterministic arrow corresponding to an ordinary arrow $p : A \times C \to B \times C$, then $\mathcal{T}^C_{A,B}(p)$ will be the nondeterministic arrow corresponding to $\mathrm{T}^C_{A,B}(p) : A \to B$. So the definition of $\mathcal{T}^C_{A,B}$ extends that of $\mathrm{T}^C_{A,B}$.

Now, given an input $a \in A$ and a target computation $u \in U$, the accessible approximation u' of u produced by p on input a will be the *feedback accessible* portion of u. This can be described as follows. Start with $c_0 = \bot_C \in C$ and $u_0 = \bot_U \in U$. For $k \geq 0$, if we have defined $c_k \in C$, then let $c_{k+1} = \pi^C_{B,C}(p^{B \times C}(\langle\langle a, c_k\rangle, u\rangle))$ and $u_{k+1} = p^U(\langle\langle a, c_k\rangle, u\rangle)$. That is, c_{k+1} and u_{k+1} are obtained at the $k+1$st stage by using as feedback input the output c_k produced at the kth stage, but using *original target computation* u. Thus u_{k+1} is the approximation of u that is accessible using as input the feedback output c_k generated at the previous stage. This procedure generates a chain $c_0 \sqsubseteq c_1 \sqsubseteq c_2 \sqsubseteq \ldots$ in C. By monotonicity, the u_k also form a chain: $u_0 \sqsubseteq u_1 \sqsubseteq u_2 \sqsubseteq \ldots$. The final accessible approximation $u' \sqsubseteq u$ is the least upper bound $\sqcup_{k \geq 0} u_k$.

Definition 5.1 is essentially the same as the definition of "feedback" that was studied in [10]. In that paper, the definition was shown to be reasonable in the sense that it gives results consistent with a more operational definition of feedback. The following result is in essence the same as Lemma 6.1 of that paper, though the axiomatic definition of "nondeterministic arrow" used in the present paper is an improvement over the more cumbersome technical setup of the previous paper, in which "objects of computations" were obtained as domains of computations of a certain kind of automaton.

Proposition 5.2 *Suppose* $p : A \times C \xrightarrow[U]{} B \times C$ *is a nondeterministic arrow. Then* $\mathcal{T}^C_{A,B}(p) : A \xrightarrow[U]{} B$ *is again a nondeterministic arrow.*

The proof of Proposition 5.2 uses the characterization $\mathcal{T}^C_{A,B}(p) = \pi^{U \times B}_{U \times B, C} \cdot (\bigsqcup_{k \geq 0} p_k)$, where $p_0 : A \times U \to U \times B \times C$ is $p \cdot \langle\langle \pi^A_{A,U}, \bot_{A \times U, C}\rangle, \pi^U_{A,U}\rangle$ and

$$p_{k+1} \quad = \quad p \cdot (1_A \times (\pi^C_{B,C} \cdot {p_k}^{B \times C}) \times 1_U) \cdot (\delta_A \times \delta_U).$$

It can be shown by induction on k that each p_k satisfies the conditions to be a nondeterministic arrow $p_k : A \xrightarrow[U]{} B \times C$, hence by continuity $\bigsqcup_{k \geq 0} p_k$ and $\mathcal{T}^C_{A,B}(p)$ do, as well.

Proposition 5.3 *For each triple of objects* (A, B, C) *the mapping* $\mathcal{T}^C_{A,B}$ *extends to a functor*

$$\mathcal{T}^C_{A,B} : \mathcal{N}\mathbf{D}(A \times C, B \times C) \to \mathcal{N}\mathbf{D}(A, C).$$

If $p : A \times C \xrightarrow[U]{} B \times C$, $q : A \times C \xrightarrow[V]{} B \times C$, *and* $\mu : p \Rightarrow q$, *then the functor takes* μ *to the morphism* $\mathcal{T}^C_{A,B}(\mu) : \mathcal{T}^C_{A,B}(p) \Rightarrow \mathcal{T}^C_{A,B}(q)$ *determined by the arrow*

$$\mu \cdot (1_A \times (\pi^C_{U \times B, C} \cdot \mathrm{T}^C_{A,B}(p)) \times 1_U) \cdot (\delta_A \times \delta_U) : A \times U \to V.$$

6 Conclusion

We have shown how, given a locally ordered bicategory $\mathbf{D}$ with finite bicategorical products, to construct a bicategory $\mathcal{N}\mathbf{D}$ of nondeterministic arrows that embeds $\mathbf{D}$ up to equivalence as a locally full subbicategory. Nondeterministic arrows satisfy some simple axioms motivated by an intuitive conception of nondeterministic computation as progressing toward an accessible approximation to a target computation. With some local completeness assumptions on $\mathbf{D}$, the bicategory $\mathcal{N}\mathbf{D}$ provides a right adjoint for each strict, sup-preserving arrow of $\mathbf{D}$; *i.e.* for each arrow of $\mathbf{D}$ that qualifies to have such an adjoint. In this case, $\mathcal{N}\mathbf{D}$ is a cartesian bicategory that is determined by its subbicategory of left adjoints (the "maps"), which is equivalent to the subbicategory of strict, sup-preserving arrows of $\mathbf{D}$. We showed how a notion of trace derived from local cpo structure on $\mathbf{D}$ extends to $\mathcal{N}\mathbf{D}$.

There are many questions that remain to be investigated. One is to see if/how cartesian-closed structure on $\mathbf{D}$ transfers to $\mathcal{N}\mathbf{D}$, thereby providing the latter with a notion of higher-order nondeterministic arrows. A general study of how local limits and/or colimits in $\mathcal{N}\mathbf{D}$ can be used to solve recursive specifications is also needed if $\mathcal{N}\mathbf{D}$ is to be of general utility as a bicategory of semantic domains. The results about trace need to be strengthened: we have only established that trace determines a family of functors on the homs of $\mathcal{N}\mathbf{D}$, but what should be shown is that $\mathcal{N}\mathbf{D}$ is in some sense a "traced monoidal bicategory." That would require a study of coherence issues for trace. It is also clear, assuming suitable completeness conditions for $\mathbf{D}$, that $\mathcal{N}\mathbf{D}$ can be described as a bicategory of fibrations in $\mathbf{D}$. A full explication of this would be helpful.

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